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Tensorial structure of a q-deformed $sp(4, R)$ superalgebra

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Abstract. A supersymmetric quantum algebra is generated by irreducible tensor operators in respect to the algebra $su_q(2)$. The even generators are realized as tensor products of q-boson creation and annihilation operators, transforming as $su_q(2)$ spinors and acting as odd generators. In this way the transformation properties of all the algebra's generators in respect to the q-deformed algebra of the angular momentum are simultaneously preserved, which is important in view of future applications in physics. In the limit $q \to 1$ the classical Lie superalgebra $sp(4. R)$ or the $osp(1|4)$ is recovered.

1. Introduction

Recently, there has been intense exploration of quantized universal enveloping algebras (QUE-algebras) so named by Drinfield [1] and Jimbo [2]. In physical applications 'quantum deformations' are very useful as they provide one or more additional parameters [3]. In their 'classical' limit to special parameters' values, the q-deformed algebras yield a conventional Lie algebra in analogy with the $\hbar \to 0$ limit in the transition from the quantum to classical mechanics. Hence quantum algebras become relevant in physics where the limits of applicability of Lie algebras are stretched. They could be used to describe perturbations from some underlying symmetry structure, which appear in many problems in statistical mechanics [4], quantum field theory [2], nuclear and molecular spectroscopy [5, 6], quantum optics [7] and so on.

In order to apply a quantum algebra in physics, a well developed theory of its representations is needed. One of the most successful steps in developing the quantum group representations was the introduction of their q-deformed oscillator realization in [8, 9]. The oscillator algebra basis in the deformed case satisfies the relations

$$a_i a_i^\dagger - q a_i^\dagger a_i = q^{-N_i} \qquad a_i a_i^\dagger - q^{-1} a_i^\dagger a_i = q^{N_i} \qquad i = 1, 2, \ldots, r \tag{1}$$

and

$$[N_i, a_i] = -a_i \qquad [N_i, a_i^\dagger] = a_i^\dagger \tag{2}$$

where the hermitian conjugate creation $a_i^\dagger$ and annihilation a_i ; $(a_i^\dagger)^\dagger = a_i$ operators are the q-analogs of boson operators and N_i are their corresponding number operators.

In the case of q not being a root of unity the properties of the quantum algebras are quite similar to those of classical Lie algebras in connection to their representation theory and, as a result, to their possible physical applications. The boson realization method was applied to construct representations of Lie algebras [10, 11], Lie superalgebras [12, 13] and loop algebras [14]. The convenience of the q-boson oscillators in the representation theory of quantum algebras naturally suggests the definition of tensor operators in terms of these oscillators [15–18]. In various physical models, especially in nuclear structure theory, irreducible tensor operators, in respect to the principal subalgebra of the angular momentum [19], are used as generators of the dynamical symmetry algebra [20–22]. This permits an easy calculation of the matrix elements of these operators in an appropriate basis by means of the Wigner–Eckart theorem.

The quantum generalization of the theory of the irreducible tensor operators and their products is reasonably well developed for the quantum algebra $U_q(su(2))$ [23–25], which can be considered as the q-analog of the quantum theory of angular momentum. In particular the Clebsh–Gordon coefficients (CGC) and the symmetry relations between them are derived in several papers (see for example [23] and the references therein). In this work the CGCs and the definition of irreducible tensor operators from [24] will be used. There the Wigner–Racah algebra for $su_q(2)$ is constructed by means of projection operators. The advantages of this approach are its independence of the explicit realization from the generators and the basis vectors and the analogy with many of the formulae of the quantum theory of angular momentum, when an ordinary c-number x is replaced by a q-number

$$[x] = \frac{q^x - q^{-x}}{q - q^{-1}} \qquad [x]_{q\to 1} \to x\,. \tag{3}$$

In view of future applications, especially in nuclear physics symmetry problems [20–22] we develop a method for constructing a supersymmetric quantum algebra, whose generators are $su_q(2)$ irreducible tensor operators. In this way the transformation properties of the creation and annihilation operators and the algebra generators with respect to the $su_q(2)$ algebra of the angular momentum are simultaneously preserved. Coming from a nuclear physics background we have turned our attention to the q-deformation of boson representations of symplectic algebras as they play the role of dynamical symmetry algebras in various phenomenological and microscopic models [20–22] developed in this field. As a first step we consider a q-deformation of the $sp(4, R)$, which by itself is of physical interest [26]. In the classical case [27] the $Sp(4, R)$ algebra already illustrates the problem of the boson realization of an $Sp(2d, R)$ Lie algebra for any integer d and for an arbitrary irrep of the $Sp(2d, R)$ group. The $sp(4, R)$ has a $u(2)$ subalgebra and thus the whole analysis can be made using the well developed Wigner–Racah algebra.

As our results are an attempt at the deformation of $sp(4, R)$, a brief review of the notations and definitions used is given in section 2. The irreducible tensor operators ITO of rank $\frac{1}{2}$ are expressed by means of two q-boson creation and annihilation operators. Then the ITOs of rank 1 are constructed as tensor products of the $su_q(2)$-spinors. Some properties and relations between the obtained ITOs are discussed in section 3. The super algebra's structure, generated by the q-tensor operators introduced, is investigated in section 4.

2. Initial definitions and notations

In this paper the Jordan–Schwinger realization [8, 9] of the quantum analogue of the $su(2)$ algebra constructed by means of two q-deformed boson operators $a_i^\dagger$ and a_i, $i = 1, 2$ which obey relations (1) and (2) is used:

$$J_+ = a_1^\dagger a_2 \qquad J_- = a_2^\dagger a_1 \qquad J_0 = \frac{N_1 - N_2}{2}\,. \tag{4}$$

These operators satisfy the commutation relations:

$$[J_0, J_\pm] = \pm J_\pm \qquad [J_+, J_-] = [2J_0] = \frac{q^{2J_0} - q^{-2J_0}}{q - q^{-1}}. \tag{5}$$

The normalized states $|jm\rangle_q$ are subsequently q-analogues of the familiar quantal angular momentum states of the basis of the finite-dimensional irreducible representation D^j. The allowed values of the 'angular momentum' are $j = 0, \frac{1}{2}, 1, \ldots$ and respectively for its 'projection': $m = j, j-1, \ldots, -j$. Hence, there is an analogy between the dimensions of the irreps D^j of $su_q(2)$ and $su(2)$: $\dim D^j = 2j + 1$.

The action of the $su_q(2)$ generators on the basis vectors gives

$$J_\pm|jm\rangle = ([j \mp m][j \pm m + 1])^{1/2}|jm \pm 1\rangle. \tag{6}$$

The generalization of the vector addition of angular momenta j_1 and j_2 to the q-deformed case, as in the 'classical' case, is defined by the expansion of the direct product of the representations $D^{j_1} \otimes D^{j_2}$ in irreducible components (the Clebsh–Gordon series). The structure of the Clebsh–Gordon series for the $su_q(2)$ algebra is the same as for the classical $su(2)$,

$$|j_2 - j_1| \leqslant j \leqslant j_1 + j_2 \qquad m = -j, -j+1, \ldots, j. \tag{7}$$

In the theory of quantum algebras the action of its generators on the direct (tensor) product of irreducible representations is given by a non-co-commutative co-product. So the components of the total angular momentum J of a quantum system consisting of two subsystems 1 and 2 are modified in the following way:

$$J_0 = J_0(1) + J_0(2) \tag{8a}$$

$$J_\pm = J_\pm(1)q^{J_0(2)} + q^{-J_0(1)}J_\pm(2). \tag{8b}$$

By definition, compatible with the formula following from the Clebsh–Gordon series , the CGCs relate vectors belonging to coupled and uncoupled bases:

$$|\, j_1 j_2; jm\rangle_q = \sum_{m_1, m_2} {}_qC^{jm}_{j_1m_1\, j_2m_2}\, |\, j_1m_1(1)\rangle|\, j_2m_2(2)\rangle. \tag{9}$$

In this paper the explicit expressions for the ${}_qC^{jm}_{j_1m_1\, j_2m_2}$ obtained in [24] are used. The CGCs are calculated as matrix elements of projection operators between vectors of the uncoupled basis (right-hand side of (9)). The projection operators are expressed in terms of powers of the generators $J_\pm$ (8b), and do not depend on their explicit representations.

The method of derivation of the CGCs insures their orthogonality and normalization. In the limit $q \to 1$ the algebraic expressions for the particular coefficients we use reduce to the corresponding standard expressions for the CGCs of $su(2)$. In our work the symmetry in relation to the sign reversal of all the projections m_1, m_2 and m,

$${}_qC^{jm}_{j_1m_1\, j_2m_2} = (-1)^{j_1+j_2-j}\, {}_{q^{-1}}C^{j-m}_{j_1-m_1\, j_2-m_2} \tag{10}$$

and the permutation symmetry property

$${}_qC^{jm}_{j_1m_1\, j_2m_2} = (-1)^{j_1+j_2-j}\, {}_{q^{-1}}C^{jm}_{j_2m_2\, j_1m_1} \tag{11}$$

are used more often.

Now the definition of the q-deformed ITO of rank l can be given. As in the 'classical case' of $su(2)$ the set of $2l+1$ components ${}_qT^l_\kappa$, $(\kappa = l, l-1, \ldots, -l)$ will be called components of a q-analog of the ITO in respect to $su_q(2)$ if the action of the group generators on the tensor operators is similar to their action on the vector states $|jm\rangle$ (6). But in the quantum case the action of the q-generators is defined by means of their co-product (8). As a result the q-tensors components obey the following commutation relations with the generators of the $su_q(2)$ algebra:

$$[J_0, T^l_\kappa] = \kappa\, T^l_\kappa \tag{12a}$$

$$J_\pm T^l_\kappa = q^\kappa T^l_\kappa J_\pm + ([l \mp \kappa][l \pm \kappa + 1])^{1/2} T^l_{\kappa\pm1} q^{-J_0}\,. \tag{12b}$$

Furthermore it is proven [7, 15, 25] that a Wigner–Eckart theorem can be formulated for these tensor operators. Thus the problem of calculating their matrix elements is reduced to calculating the reduced matrix elements $\langle j' \| T^l \| j'' \rangle$ with coefficient proportional to the CGCs.

After formulating the 'quantum' tensor operator T^l_κ the $su_q(2)$ basis' vectors $T^l_\kappa|jm\rangle$, which transform according to the direct product $D^j \otimes D^l$ can be presented as linear combinations by means of the CGCs of the vector components $|rs\rangle$ transforming according to the irrep D^r:

$${}_qT^l_\kappa|jm\rangle = \sum_{r,s} {}_qC^{rs}_{jm\,l\kappa}\,|l\,j\,;\,rs\rangle\,. \tag{13}$$

Using this type of 'mapping' it is easy to prove that there exists an algebra of q-tensor operators generated by tensor products of ITOs, which are by themselves irreducible tensor operators [15]:

$${}_q\{T^l \otimes T^j\} = \sum_{r,s} {}_qC^{rs}_{l\kappa\, jm}\, T^r_s\,. \tag{14}$$

We are now ready to investigate what type of tensor algebras can be generated by the two q-boson creation and annihilation operators $a^\dagger_i; a_i$ $i = 1, 2$ (1), which are used for the Gordon–Schwinger realization of the $su_q(2)$.

3. Construction of q-tensor operators and their properties

The $su_q(2)$ algebra (5) can be generated by the operators (4) expressed in terms of q-boson creation and annihilation operators $a^\dagger_i, a_i$, $i = 1, 2$. These operators by themselves close a deformation of the Heisenberg algebra—(1) and (2). We would like to clarify the transformation properties of the creation and annihilation operators in respect to the $su_q(2)$ representations. As a first example of a q-tensor operator Biedenharn [15] defines the pair of creation operators

$$t_{\frac12,\frac12} = a^\dagger_1 q^{(N_2)/2} \qquad t_{\frac12,-\frac12} = a^\dagger_2 q^{-N_1/2}\,. \tag{15}$$

Thus (15) is a q-tensor operator of rank $\frac12$ as its two components obey the conditions (12a, b) for $l = \frac12$. It is easy to see that the pair of annihilation operators

$$\tilde{t}_{\frac12,\frac12} = q^{-1/2} a_2 q^{-N_1/2} \qquad \tilde{t}_{\frac12,-\frac12} = -q^{1/2} a_1 q^{N_2/2} \tag{16}$$

satisfy the same conditions (12a, b) thus representing another q-tensor operator $\tilde{t}_{\frac{1}{2},\alpha}$. From the hermitian conjugation of the creation and annihilation operators follow the conjugation relations between the two q-spinors:

$$(t_{k,\alpha})^{\dagger} = (-1)^{(k-\alpha+1)}\, q^{-\alpha}\tilde{t}_{k,-\alpha} \qquad (\tilde{t}_{k,\alpha})^{\dagger} = (-1)^{(k-\alpha)}\, q^{-\alpha} t_{k,-\alpha}\,. \tag{17}$$

They are exactly the same as for the usual $su(2)$-spinors, when $q \to 1$. With the two q-tensor operators of rank $\frac{1}{2}$, $t_{1/2}$ and $\tilde{t}_{1/2}$, three different tensor products can be constructed. Each one can be co-multiplied by itself:

$$\{t_{\frac{1}{2}} \otimes t_{\frac{1}{2}}\} = \bigoplus_{l,m} T^{l}_{m} \qquad l = 0, 1 \quad m = -l, -l+1, \ldots, l \tag{18}$$

and

$$\{\tilde{t}_{\frac{1}{2}} \otimes \tilde{t}_{\frac{1}{2}}\} = \bigoplus_{l,m} \tilde{T}^{l}_{m} \qquad l = 0, 1 \quad m = -l, -l+1, \ldots, l\,. \tag{19}$$

Inserting in (14) the necessary $_q$CGCs, calculated from the tables in [24], the explicit representations of the components of the q-tensors T^1_μ and $\tilde{T}^1_\mu$ in terms of q-bosons $a_i^{\dagger}$ and a_i; $i = 1, 2$ is obtained:

$$\begin{aligned} &T^1_1 = (a_1^{\dagger})^2 q^{N_2} \qquad T^1_{-1} = (a_2^{\dagger})^2 q^{-N_1} \qquad T^1_0 = \sqrt{[2]}\, a_1^{\dagger} a_2^{\dagger}\, q^{-J_0} \\ &\tilde{T}^1_1 = q^{-1}(a_2)^2 q^{-N_1} \qquad \tilde{T}^1_{-1} = q^{1}(a_1)^2 q^{N_2} \qquad \tilde{T}^1_0 = -\sqrt{[2]}\, a_1 a_2\, q^{-J_0}\,. \end{aligned} \tag{20}$$

The tensors carry the scalar representations in (18) and (19) and $T^0_0 = 0$ and $\tilde{T}^0_0 = 0$ as a result of (1) and the CGCs symmetries (11). The conjugation relations for the operators (20) read

$$(T^l_\kappa)^{\dagger} = (-1)^{(l-\kappa)} q^{-\kappa} \tilde{T}^l_{-\kappa} \tag{21}$$

$$(\tilde{T}^l_\kappa)^{\dagger} = (-1)^{(l-\kappa)} q^{-\kappa} T^l_{-\kappa}\,. \tag{22}$$

The third possible q-tensor product is obtained by multiplication of the two conjugated q-spinors:

$$\{t_{\frac{1}{2}} \otimes \tilde{t}_{\frac{1}{2}}\} = \bigoplus_{l,m} L^{l}_{m} \qquad l = 0, 1 \quad m = -l, -l+1, \ldots, l\,. \tag{23}$$

The three-dimensional q-tensor operator L^1_m $m = 0, \pm 1$ has the explicit realization, in terms of q-bosons,

$$\begin{aligned} &L^1_1 = q^{-1} a_1^{\dagger} a_2 q^{-(N_1+N_2)/2} = q^{-1} J_+ q^{-J_0} \qquad L^1_{-1} = -q a_2^{\dagger} a_1 q^{-(N_1+N_2)/2} = -q J_- q^{-J_0} \\ &L^1_0 = \frac{1}{\sqrt{[2]}}\big(q^{-1} a_2^{\dagger} a_2 q^{-N_1} - q a_1^{\dagger} a_1 q^{N_2}\big) = \frac{1}{\sqrt{[2]}}\big(q^{-1}[N_2] q^{-N_1} - q[N_1] q^{N_2}\big) \\ &\quad = \frac{1}{\sqrt{[2]}}\big(q^{-1} J_- J_+ - q J_+ J_-\big) = -\frac{1}{\sqrt{[2]}}\big(q^{-1}[2J_0] + (q - q^{-1}) J_+ J_-\big)\,. \end{aligned} \tag{24}$$

The components of this operator actually represent $su_q(2)$ generators (5) as irreducible q-tensor operators. This tensor product also carries a one-dimensional scalar representation

$$L_0^0 = \frac{1}{\sqrt{[2]}}\left(a_2^{\dagger} a_2 q^{-N_1} + a_1^{\dagger} a_1 q^{N_2}\right) = \frac{[N]}{\sqrt{[2]}} \tag{25}$$

where $N = N_1 + N_2$ is the total number of bosons. It should be noticed that the decomposition (23) is obtained from $\{t_{\frac{1}{2}} \otimes \tilde{t}_{\frac{1}{2}}\}$, e.g.

$$L_m^1 = \sum_{\mu,\nu} {}_q C^{1m}_{\frac{1}{2}\mu\,\frac{1}{2}\nu} t_{\frac{1}{2}\nu} \tilde{t}_{\frac{1}{2}\mu} = \sum_{\mu,\nu} {}_q C^{1m}_{\frac{1}{2}\mu\,\frac{1}{2}\nu} \tilde{t}_{\frac{1}{2}\nu} t_{\frac{1}{2}\mu} \,. \tag{26}$$

The conjugation relations for the components of the operator L_m^1 which follow from (1) and the symmetry properties (10) and (11) of the CGCs are

$$(L_m^1)^{\dagger} = (-1)^{-m} q^{-m} L_{-m}^1 \tag{27}$$

which contracts to the conjugation relation of irreducible tensor operator of rank 1 of $su(2)$ or the Cartan components of the angular momentum operator, when $q \to 1$. With the help of the relations (1) and (2) and the q-boson realization (4) of the $su_q(2)$-generators it can be verified that all three tensors T_m^1 , $\tilde{T}_m^1$ (21) or L_m^1 (24) are of rank 1 since each ones' three components satisfy the definition (12) for $l = 1$.

At the end of this section two more relations are presented, which are derived from (24) and (25) and are used in the following calculations:

$$\begin{aligned} L_0^1 + q L_0^0 &= \sqrt{[2]}\,[N_2] q^{-N_1} = \frac{\sqrt{[2]}}{(q - q^{-1})}\,(q^{-2J_0} - q^{-N}) \\ L_0^1 - q^{-1} L_0^0 &= -\sqrt{[2]}\,[N_1] q^{N_2} = \frac{\sqrt{[2]}}{(q - q^{-1})}\,(q^{-2J_0} - q^{N})\,. \end{aligned} \tag{28}$$

From (28) the operators q^N and q^{-N} can be expressed by means of the $m = 0$ components of the $L_m^l; l = 0, 1$ operator and the operator q^{-2J_0}.

4. Algebraic structure generated by the tensor operators

In this approach we create a q-deformed algebra by preserving the transformation properties in relation to the $su_q(2)$ of the basic oscillators and their tensor products, which play the role of generators. Thus an algebra of q-tensor operators is generated by their components. Usually, when an algebra is deformed, it is difficult to see what type of q-commutators are satisfied by the generators of the q-deformed algebra. In [28] a q-deformed superalgebra is closed by satisfying usual commutation and anticommutation relations [28] between the components of the q-tensor operators of rank $\frac{1}{2}$ and 1. The even part is a boson representation of a q-deformation of the $Sp(4, R)$ algebra which is enlarged with one more generator–q^{-2J_0}. It commutes with all the other generators in the following way:

$$q^{-2J_0} I_m^l = q^{-2m} I_m^l q^{-2J_0} \tag{29}$$

where I_m^l is any of the fourteen components of the operators (15), (16), (20), (24) and (25). But this is a kind of a spurious generator, since in the limit $q \to 1, q^{-2J_0} \to 1$ the classical boson representation of the $sp(4, R)$ algebra is recovered. Moreover it is easy to cancel it in the so-constructed superalgebra, if all the fourteen even and odd generators are rescaled in the following way:

$$t_{\frac{1}{2},m} = f_{\frac{1}{2},m}\, q^{-J_0} \quad m = \pm\tfrac{1}{2} \qquad \tilde{t}_{\frac{1}{2},m} = g_{\frac{1}{2},m}\, q^{-J_0} \quad m = \pm\tfrac{1}{2} \tag{30a}$$

$$\begin{aligned} T^l_m &= \sqrt{[1+\delta_{m,0}]}F^l_m\, q^{-2J_0} && l=1 \quad m=0,\pm 1 \\ \tilde{T}^l_m &= \sqrt{[1+\delta_{m,0}]}G^l_m\, q^{-2J_0} && l=1 \quad m=0,\pm 1 \\ L^l_m &= \sqrt{[1+\delta_{m,0}]}A^l_m\, q^{-2J_0} && l=0,1 \quad m=0,\pm 1\,. \end{aligned} \tag{30b}$$

Obviously, as a result of (30) all the usual commutators and anticommutators in [28] transform to q-commutators and q-anticommutators:

$$[S^{l_1}_{m_1}, S'^{l_2}_{m_2}]_{\pm q^\nu} = S^{l_1}_{m_1} S'^{l_2}_{m_2} \pm q^\nu S'^{l_2}_{m_2} S^{l_1}_{m_1} \tag{31}$$

where $S_{\frac{1}{2},m}$, $S'_{\frac{1}{2},m}$ are any of the operators from (30) and ν depends on the projection indexes m_1, m_2 of the two ITOs taking part in them. There is also a minor change in the rescaled operators defining properties as ITOs. The first term in (12b) is multiplied by $q^{\mp 2}$ for the the rank 1 tensors (30b) and by $q^{\mp 1}$ for the rank $\frac{1}{2}$ tensors.

As the building blocks of the algebra investigated are the spinor operators (15) and (16), their rescaled components from (30a) are considered as odd generators. The q-anticommutators of the odd generators, calculated with the use of the basic relations (1) and (2) between the q boson creation and annihilation operators, are

$$\begin{aligned} \{f_{\frac{1}{2},m_1}, f_{\frac{1}{2},m_2}\}_{q^\nu} &= q^{(m_1+m_2)/2}(1+q^{m_2-m_1})\, F^1_{m_1+m_2} \\ \{g_{\frac{1}{2},m_1}, g_{\frac{1}{2},m_2}\}_{q^\nu} &= q^{(m_1+m_2)/2}(1+q^{m_2-m_1})\, G^1_{m_1+m_2} \\ \{f_{\frac{1}{2},m_1}, g_{\frac{1}{2},m_2}\}_{q^\nu} &= q^{(m_1+m_2)/2}(1+q^{m_2-m_1})(A^1_{m_1+m_2} + 2m_2\,\delta_{m_1,-m_2}\, q^{2m_2}\, L^0_0) + 2m_2\delta_{m_1,-m_2} \end{aligned} \tag{32}$$

where $\nu = m_2 - m_1$. Hence the anticommutators of the odd generators, as in the classical case, behave as a representation of the even part of the q-deformed super algebra. The last equation in (32) naturally suggests the introduction of a combination of the $m = 0$ components of the A^l_m; $(l = 0, 1)$ operators. Hence, instead of the rescaled standard cyclic components of the angular momentum operator (30b), which commute in the following way:

$$[A^1_{m_1}, A^1_{m_2}]_{q^{2(m_2-m_1)}} = [m_1 - m_2] q^{-(m_1-m_2)}\, A^1_{m_1+m_2} \tag{33}$$

we will use the combinations like (28) denoted as operators with two 'projection' indices

$$A_{m,\mu} = A^1_m + \delta_{m,0}\, q^\mu\, A^0_0 \qquad \begin{cases} \mu = 0 & \text{when } m = \pm 1 \\ \mu = \pm 1 & \text{when } m = 0\,. \end{cases} \tag{34}$$

The A^0_0 operator is an invariant of this algebra, (33), because it commutes with the operators A^1_m:

$$[A^1_m, A^0_0] = 0\,. \tag{35}$$

In all the commutators between the even generators the power of the defining term q^{ν_e} is $\nu_e = 2(m_2 - m_1)$ as in (33).

In the previous section an other two q-tensor operators T^1_m and $\tilde{T}^1_m$, carrying the three-dimensional $m = 0, \pm 1$ irreducible representation of $su_q(2)$ were constructed. They differ from the L^1_m operator, discussed above, in the commutation relations between their components. The relevant relations between their rescaled (30*b*) counterparts are

$$\begin{aligned} [F^1_{m_1}, F^1_{m_2}]_{q^{\nu_e}} &= [m_1 - m_2] q^{-(m_1-m_2)} \lambda F^1_0 F^1_{m_1+m_2} \\ [G^1_{m_1}, G^1_{m_2}]_{q^{\nu_e}} &= [m_1 - m_2] q^{-(m_1-m_2)} \lambda G^1_0 G^1_{m_1+m_2} \end{aligned} \tag{36}$$

where $\lambda = (q - q^{-1})$. For $q \to 1$ the components of these q-tensors commute.The tensors F^1_m and G^1_m have the structure of raising and lowering q-deformed pair operators (20) and A^1_m (24) and A^0_0 (25) of the q-deformed multipole operators. The commutation relations (33) and (36) between the components of the three first-rank tensors close into themselves, so generating three q-deformed subalgebras. The q-commutators (36), which $\to 0$ in the classical limit of $q \to 1$ are now quadratic and proportional to the product of the $m = 0$ component with the component with the correct projection, according to the CG expansion (14), of the relevant tensor. The number of the components of the q-tensors introduced ((20), (24) and (25)) is exactly 10, which is the number of the $sp(4, R)$ generators. In order to close such a type of a larger algebra, the q-commutation relations between the subalgebra's generators (33) and (36) have to be calculated. We start with the commutators between the raising, F^1_m, and lowering G^1_m, pair, operators:

$$[F^1_{m_1}, G^1_{m_2}]_{q^{2(m_2-m_1)}} = [m_1 - m_2] q^{-(m_1-m_2)} \{(2 - \lambda \mathcal{A}_{0,-m_1}) \mathcal{A}_{m_1+m_2,m_2} + m_2 q^{m_2}\} \quad m_1 \neq 0 \tag{37}$$

$$[F^1_0, G^1_{m_2}]_{q^{2m_2}} = q^{m_2} \mathcal{A}_{m_2,0}(2 - \lambda \mathcal{A}_{0,m_2}) \quad m_2 \neq 0$$

$$[F^1_0, G^1_0] = 1 + q^{-1} \mathcal{A}_{0,1} - q \mathcal{A}_{0,-1} .$$

These commutators are also with a quadratic term proportional to the components of the angular momentum operator in a way which gives, in the limit $q \to 1$, the relevant commutation relations of the classical Lie algebra of $Sp(4, R)$. Furthermore, the commutators between the q-deformed multipole operators $\mathcal{A}_{m,\mu}$, $(m = 0, \pm 1;\ \mu = \pm 1, 0)$ (34) and the pair operators F^1_m and G^1_m for $m = 0, \pm 1$ are proportional to the components of the pair operators taking part in them, respectively:

$$\begin{aligned} [\mathcal{A}_{m_1,\mu}, F^1_{m_2}]_{q^{\nu_e}} &= [m_1 - m_2 + \mu] q^{-(m_1-m_2+\mu)} F^1_{m_1+m_2} \\ [\mathcal{A}_{m_1,\mu}, G^1_{m_2}]_{q^{\nu_e}} &= [m_1 - m_2 + \mu] q^{-(m_1-m_2-\mu)} G^1_{m_1+m_2} . \end{aligned} \tag{38}$$

Equation (38) are easily calculated using the q-tensorial definition (20) of the components of the T^1_m and $\tilde{T}^1_m$ operators and the representation of $L^1_{\pm 1}$ (24) in terms of the $su_q(2)$ generators $J_\pm$ respectively. In the limit $q \to 1$ (36)–(38) correctly reproduce the relevant commutation relations of the tensor representation of the classical $Sp(4, R)$. In summary, the nine components of the rescaled (30*b*) three q-tensor operators of rank $1 - T^1_m$, $\tilde{T}^1_m$ and L^1_m; $(m = 0, \pm 1)$ and the scalar L^0_0 commute as generators of a q-deformed algebra which in the limit $q \to 1$ reproduces the commutation relations of the generators of the Lie algebra of $Sp(4, R)$ in terms of irreducible $su(2)$ tensor operators. The quantum algebra constructed in this way represents the even part of a q-deformed superalgebra.

In order to close the superalgebra we have to determine the commutation relations between the even and odd generators. In this case they are determined by the power $\nu_{eo} = 2m_2 - m_1$. We start with the q-tensor $f_{\frac{1}{2},\alpha}$ and its co-product F^1_m:

$$[F^1_{m_1}, f_{\frac{1}{2},m_2}]_{q^{\nu_{eo}}} = q^{-m_1/2}\,(1 - q^{2m_2-m_1})\, F^1_0\, f_{\frac{1}{2},m_1+m_2}\,. \tag{39}$$

In the limit $q \to 1$ all the commutators (39) are 0.

The q-commutators $[G^1_m, g_{\frac{1}{2},\alpha}]_{q^{\nu_{eo}}}$ for $m = 0, \pm 1$ and $\alpha = \pm\frac{1}{2}$ coincide with (39) when replacing the $\{F\}$, $\{f\}$ operators with their conjugated counterparts (17) and (21).

$$[G^1_{m_1}, g_{\frac{1}{2},m_2}]_{q^{\nu_{eo}}} = q^{-m_1/2}\,(1 - q^{2m_2-m_1})\, G^1_0\, g_{\frac{1}{2},m_1+m_2}\,. \tag{40}$$

The mixed commutators of $\{F\}$ with $\{g\}$ are as follows:

$$[F^1_{m_1}, g_{\frac{1}{2},m_2}]_{q^{\nu_{eo}}} = q^{-m_1/2}\{(1 - q^{2m_2-m_1})\mathcal{A}_{0,2m_2} + (m_1 - 2m_2)\, q^{2m_2}\} f_{\frac{1}{2},m_1+m_2}\,. \tag{41}$$

By conjugating (41) and moving the resulting $g_{\frac{1}{2},\alpha}$, $\alpha = \pm\frac{1}{2}$ component to the right-hand side we obtain

$$[G^1_{m_1}, f_{\frac{1}{2},m_2}]_{q^{\nu_{eo}}} = q^{m_1/2}\, q^{2(m_2-m_1)}\{(1 - q^{2m_2-m_1})\,\mathcal{A}_{0,-2m_2} - [m_1 - 2m_2]\} g_{\frac{1}{2},m_1+m_2}\,. \tag{42}$$

At the end the commutators of the multipole q-operator are $\mathcal{A}_{m,\mu}$ with the two q-spinors are

$$\begin{aligned}
[\mathcal{A}^1_{m_1}, f_{\frac{1}{2},m_2}]_{q^{\nu_{eo}}} &= \{q^{m_2-m_1+\frac{\mu}{2}}(m_1 - 2m_2) + \delta_{m_1,0}(q^{m_1} - q^{\mu})\mathcal{A}_{0,\mu}\} f_{\frac{1}{2},m_1+m_2} \\
[\mathcal{A}^1_{m_1}, g_{\frac{1}{2},m_2}]_{q^{\nu_{eo}}} &= \{q^{m_2-m_1-\frac{\mu}{2}}(m_1 - 2m_2) + \delta_{m_1,0}(q^{m_1} - q^{-\mu})\mathcal{A}_{0,-\mu}\} g_{\frac{1}{2},m_1+m_2}\,.
\end{aligned} \tag{43}$$

Thus a q-deformation of a superalgebra, defined by the relations (32)–(43) is obtained. All the commutation relations, except those between the odd generators (32), between the cyclic components of the angular momentum operator (33) and between the multipole and pair operators (38) are quadratic. The even part itself becomes a quadratic $sp_q(4, R)$ algebra [29]. As a result a q-deformation of the oscillator realization of the $sp(4, R)$ is obtained, which is an embedding of a representation of the $sp_q(4, R)$ superalgebra in the infinite-dimensional associative superalgebra of the $su_q(2)$-spinors . In the limit $q \to 1$ the deformed superalgebra contracts to the $sp(4, R)$ superalgebra. The algebra is realized by co-multiplication of boson creation and annihilation operators only, with well defined transformation properties with respect to the irreducible representations of its subalgebra $su_q(2)$. The co-multiplication preserves the commutation relations, which permits the use of direct products of representations in the construction [30]. Hence the algebra generators are q-tensor operators, which makes the subsequent investigation of the vector representations and the calculation of the matrix elements of the generators easier. The generalization of such a realization of the q-deformed $sp(2n, R)$ for $n = 1, 2, \ldots$ is possible [31].

Usually the quantization of a simple Lie algebra [10, 11] or Lie superalgebra [12, 13] is realized in terms of its Chevalley generators. Such a deformation of the $osp(1, 2n)$ (or $B(0, n)$ in Kac's notation) is realized in terms of creation and annihilation operators of n ordinary harmonic oscillators in [13]. In our work the whole Cartan–Weyl basis of a real form of the q-deformed $osp(1, 4)$ algebra is obtained (for q real). We use the notation $sp(4, R)$ superalgebra as it is more popular in physical applications and stresses that the even part $sp(4, R)$ is a subalgebra, which can be investigated and applied independently. These type of algebras are of physical interest for application in nuclear structure models.

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